The quantum mechanics of the ideal asymmetric top with spin (frozen Mixmaster universe)

1974 J. Phys. A: Math. Nucl. Gen. 7 1527

(http://iopscience.iop.org/0301-0015/7/13/007)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 13, 1974. Printed in Great Britain. © 1974

The quantum mechanics of the ideal asymmetric top with spin

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Received 14 February 1974

Abstract. The body-fixed symmetry of the ideal asymmetric top is that of the double four (or quaternion) group. For integral spin this is equivalent to the ordinary four-group and the secular determinant breaks down diagonally into four distinct blocks, as in the standard theory. For half-odd-integral spin, however, we obtain two equivalent blocks, the levels being doubly degenerate. A generalization is then presented which entails making the top wavefunction a spinor-valued quantity. This is expanded in spinor hyperspherical harmonics which are generalizations of the ordinary hyperspherical harmonics, the $\mathscr{D}^{L}_{MN}$. A systematic procedure is given for finding the elements of the secular determinant. The motivation for the generalization is that of spinor fields in a frozen Mixmaster universe.

1. Introduction

The problem of the asymmetric top in quantum mechanics is a standard one, dealt with by a number of authors (eg Landau and Lifshitz (1965); for a review of the topic see van Winter (1954); the work of King *et al* (1943) should also be mentioned). New interest in it has arisen following the recent publication of a paper (Hu 1973) in which the problem of scalar waves in a frozen Mixmaster universe (see Misner 1969) is shown to be mathematically identical to that of the asymmetric top, except that half-integral angular momenta are allowed as well as integral. We call this the ideal asymmetric top. Hu, however, does not deal with these half-integral spins as he explains in a later paper (Hu *et al* 1973), saying that they lead to complications, and so this is the first thing to be investigated, in § 2. It will be seen that there is a convenient, though different, symmetry decomposition and the situation turns out not to be so complicated after all. The theory of the ideal *spherical* top has been discussed by Schulman (1968).

If we take, instead of scalar waves, the case of spinor waves in an anisotropic universe —a problem which is equally as interesting—then Hu's work must be generalized. The wave equation for the (massive) $(2j+1)$-spinor ϕ in curved space can be written as (Dowker 1967)

$$(\nabla_\mu \nabla^\mu + kR_{\mu\nu\sigma\rho}j^{\mu\nu}j^{\sigma\rho} - m^2)\phi = 0 \tag{1}$$

where k is a constant for a given spin. The $j^{\mu\nu}$ are the generators of the homogeneous Lorentz group in the $(j, 0)$ representation and $R_{\mu\nu\sigma\rho}$ is the curvature tensor. All of these quantities are known for the case of the frozen Mixmaster universe as dealt with by Hu, to whom the reader is referred for the appropriate definitions. $R_{\mu\nu\sigma\rho}$ can be calculated using the calculus of Cartan frames (see eg Misner 1963, appendix A), and it is found that

$$R_{\mu\nu\sigma\rho}j^{\mu\nu}j^{\sigma\rho} = \sum_i a_i j_i^2. \tag{2}$$

Here the a_i are constants and the j_i are the standard spin-j angular momentum operators. We may use, for the spatial part of the covariant derivative (Dowker 1972)

$$\nabla_i = b_i L_i + c_i j_i \tag{3}$$

where the slightly more complicated form arises from the more general shape of this particular universe. The calculation of the covariant derivative is also most conveniently performed using Cartan frames (see eg Brill and Cohen 1966 for an example).

We define $\boldsymbol{J} = \boldsymbol{j} + \boldsymbol{L}$ as the 'total angular momentum' and separate out the time dependence of (1) to obtain, using (2) and (3), an equation of the form

$$\left(\sum_i (l_i L_i^2 + m_i j_i^2 + n_i J_i^2) + (m^2 - E^2)\right)\psi = 0 \tag{4}$$

where ψ is a time independent $(2j+1)$-spinor. Equation (4) may be viewed as describing a top 'with spin j' and of 'energy' $E^2 - m^2$ where the 'hamiltonian' will be given by

$$\mathscr{H} = \sum_i (l_i L_i^2 + m_i j_i^2 + n_i J_i^2). \tag{5}$$

The subject of § 3 will be the calculation (at least in principle) of the eigenvalues of this hamiltonian and this will enable the problem of spinor waves in a frozen Mixmaster universe to be solved completely, although the computation of the eigenfunctions is not actually performed here.

We should point out that the problem of massless fields also reduces to an analysis of the hamiltonian (5).

2. The asymmetric top for integral and half-integral spin. The 'ideal' top

We begin by considering the symmetry properties of the top. Rotations of π about any of the three major axes carry the top into a configuration that cannot be distinguished from the original one. We shall call these three operators C^a, C^b, C^c. The hamiltonian of the top is of the form

$$\mathscr{H}_0 = aL_1^2 + bL_2^2 + cL_3^2 \tag{6}$$

where a, b, c are related to the moments of inertia of the top, and L_i are the components of the angular momentum operator L along the three principal axes of inertia of the top. Thus the L_i are a body-fixed set. Plainly the hamiltonian commutes with the C.

Now we proceed in a standard way and expand the wavefunction of the asymmetric top in terms of suitable basis functions. The set that springs to mind is that of the symmetric-top wavefunctions, which are just the representation functions of SU(2), and the expansion is of the form

$$|JM\rangle = \sum_K a_K^J |JMK\rangle$$

where, if q are coordinates, the Euler angles α, β, γ say,

$$\langle q|JMK\rangle \equiv \mathscr{D}^J_{MK}(\alpha\beta\gamma)$$

in the standard fashion. The $\mathscr{D}^J_{MK}$ are of course well known. We note that the body-fixed set of operators L_i operate only on the right index of the $\mathscr{D}^J_{MK}$, the left index being operated on by space-fixed operators $\tilde{L}_a$. In the absence of external fields etc, the asymmetric top retains a $(2J+1)$ degeneracy, corresponding to the possible values of the left index which indicates the orientation, with respect to space-fixed axes, of the spinning top.

The effect of the above symmetry operations on our basis functions may be found, thus,

$$\begin{aligned} C^a|JNM\rangle &= \sum_K \mathscr{D}^J_{MK}(0\pi 0)|JNK\rangle = \mathrm{e}^{-\mathrm{i}(J+M)\pi}|JN-M\rangle \\ C^b|JNM\rangle &= \sum_K \mathscr{D}^J_{MK}(\pi\pi 0)|JNK\rangle = \mathrm{e}^{\mathrm{i}J\pi}|JN-M\rangle \\ C^c|JNM\rangle &= \sum_K \mathscr{D}^J_{MK}(\pi 00)|JNK\rangle = \mathrm{e}^{-\mathrm{i}M\pi}|JNM\rangle. \end{aligned} \tag{7}$$

(We have used the conventions of Brink and Satchler (1962).)

If J is integral then C^a, C^b, C^c intercommute and there exists therefore a set of simultaneous eigenfunctions of them. If these eigenfunctions are used instead of the above, as a basis, then a considerable simplification results.

The required eigenfunctions are given by

$$|JNM\gamma\rangle = (2)^{-1/2}(|JNM\rangle + (-1)^{\gamma}|JN-M\rangle) \tag{8}$$

where $\gamma = 0, 1$. From (7) and (8) we obtain

$$\begin{aligned} C^a|JNM\gamma\rangle &= (-1)^{\gamma+J+M}|JNM\gamma\rangle \\ C^b|JNM\gamma\rangle &= (-1)^{\gamma+J}|JNM\gamma\rangle \\ C^c|JNM\gamma\rangle &= (-1)^{M}|JNM\gamma\rangle. \end{aligned}$$

These three operators, plus the identity, form the four-group D_2 (Landau and Lifshitz 1965, p 355). This has four irreducible representations, all of them one dimensional, and the parities of M and $(J+\gamma)$ label the particular representation of D_2 to which the eigenfunctions belong (table 1).

Table 1. Irreducible representations A, B^a, B^b, B^c of the group D_2.

	A	B^a	B^b	B^c
1	+	+	+	+
C^a	+	+	−	−
C^b	+	−	+	−
C^c	+	−	−	+

The important result is that since the hamiltonian $\mathscr{H}_0$, equation (6), commutes with all of the C and hence with all of the elements of an irreducible representation of D_2, then $\mathscr{H}_0$ must be a multiple of the unit element, by Schur's lemma. Therefore $\mathscr{H}_0$ cannot connect states belonging to different representations of the group D_2, ie we must have that, for the matrix elements $[\mathscr{H}_0]^{\gamma'\gamma}_{M'M} = \langle JNM'\gamma'|\mathscr{H}_0|JNM\gamma\rangle$,

$$(-1)^{\gamma'+J} = (-1)^{\gamma+J}$$

and

$$(-1)^{M'} = (-1)^{M}.$$

Thus the energy matrix immediately breaks into four blocks, which with suitable labelling lie along the diagonal. To each block there correspond the parities of $(\gamma + J)$ and M.

All of this is nothing new; but suppose now that J is half integral. The C then no longer commute (in fact they anticommute) and so there do not exist simultaneous eigenfunctions of them. However, the elements ± 1, $\pm C^a$, $\pm C^b$, $\pm C^c$ together make up the eight-group (quaternion group) D_2', which has but one faithful two-dimensional representation (Landau and Lifshitz 1965, p 370)—a good example is $\pm\sigma_i$, ± 1 where σ_i are the Pauli matrices. By the same reasoning as before, $\mathscr{H}_0$ is a multiple of the unit element which means in this case that every energy level is doubly degenerate and hence that, with a suitable choice of basis, the hamiltonian matrix will factorize into two matrices of equal determinant. This is fairly easy to prove, but for now the fact will just be stated. In § 3 the proof will be given for a more general case from which this case will be deducible.

All that remains is the calculation of the matrix elements $[\mathscr{H}_0]^{\gamma'\gamma}_{M'M}$. Again this is very easy since the elements $\langle JNM'|\mathscr{H}_0|JNM\rangle$ are known (Landau and Lifshitz 1965, p 387). Thus, in principle, the problem of the asymmetric top for integral or half-integral spin is solved.

There is just one more point concerning the form of the hamiltonian (5) and its matrix. We may write, according to Brink and Satchler (1962, p 100), and using their notation,

$$aL_1^2 + bL_2^2 + cL_3^2 = \alpha_L T_0^0(\boldsymbol{L}, \boldsymbol{L}) + \beta_L T_0^2(\boldsymbol{L}, \boldsymbol{L}) + \gamma_L[T_2^2(\boldsymbol{L}, \boldsymbol{L}) + T_{-2}^2(\boldsymbol{L}, \boldsymbol{L})], \tag{9}$$

where the T are tensor (product) operators and α_L, β_L, γ_L are given in terms of a, b, c. Nonzero matrix elements are obtained only if

$$M' \pm M = 0, \pm 2$$

and

$$\Delta(J, J, 2),$$

which is satisfied identically.

Thus, with suitable labelling, our matrices can be written as tri-diagonal ones, the determinants of which can be expanded in terms of continued fractions. There is then a very convenient iterative method of computing the eigenvalues (King *et al* 1943).

The eigen-energies for tops with a variety of half-integral spins have been calculated. For comparison with Hu (1973) we have given in figure 1 graphs of energy against 'shape and deformation' parameters α, β for a few spins, where α and β are related to a, b, c above by $a = 1/2l_1^2$, $b = 1/2l_2^2$, $c = 1/2l_3^2$ and

$$l_1 = l_0 \exp[\beta \cos(\alpha - \pi/3)]$$

$$l_2 = l_0 \exp[\beta \cos(\alpha + \pi/3)]$$

$$l_3 = l_0 \exp[-\beta \cos \alpha],$$

where l_0 is a constant. These parameters are related to the shape of the frozen Mixmaster universe, mentioned above.

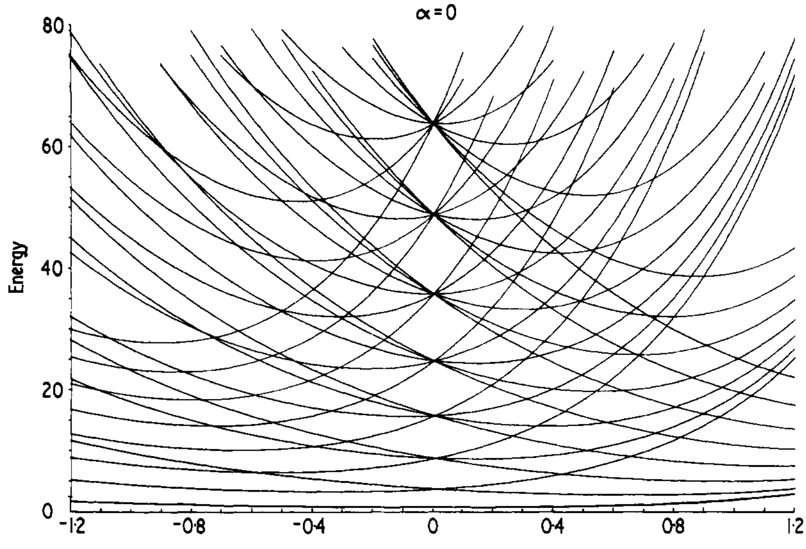

Figure 1. Energies of scalar waves in the Mixmaster universe for values $J = \frac{1}{2}, \ldots \frac{13}{2}$ plotted against shape and deformation parameters α, β,

3. Top with spin

The theory of the asymmetric top outlined in the previous section has been used (Hu 1973, Hu *et al* 1973) in the discussion of scalar fields in an expanding universe. If we wish to consider higher-spin fields then the theory must be generalized and leads to what we term the quantum mechanics of an asymmetric top 'with spin'. The wave-function now possesses a spinor index μ, ranging from $-j$ to $+j$, and takes the form $\psi_\mu(q) = \langle \mu, q|\psi\rangle$. The operator hamiltonian for the top with spin is

$$\mathscr{H} = \sum_i (l_i L_i^2 + m_i j_i^2 + n_i J_i^2), \tag{10}$$

where $\boldsymbol{J} = \boldsymbol{j} + \boldsymbol{L}$ is the 'total' angular momentum (equivalently, the generalized Lie operator) obtained by combining the 'orbital' angular momentum $\boldsymbol{L}$ with the intrinsic spin $\boldsymbol{j}$. When the l_i, m_i, n_i are equal to l, m, n respectively, we call the system a *spherical* top with spin, for which $\mathscr{H} = \mathscr{H}_s \equiv l\boldsymbol{L}^2 + m\boldsymbol{j}^2 + n\boldsymbol{J}^2$. An example of this occurs in the theory of higher-spin particles in an Einstein universe, the spatial part of which is a three-sphere. The field equations involve the 'spinor laplacian' (Dowker 1972),

$$\Delta = (\boldsymbol{X} + \boldsymbol{\Gamma}) \cdot (\boldsymbol{X} + \boldsymbol{\Gamma}),$$

which is a special spherical case of (10). Here X_i are the right generators of motions of the three-sphere into itself, that is, of the group SU(2), and are therefore the angular momentum operators L_i of the top; and the Γ_i are proportional to the spin angular momentum operators j_i. A complete set of commuting operators for Δ (and for any $\mathscr{H}_s$) is $\boldsymbol{L}^2$ $(\equiv \tilde{\boldsymbol{L}}^2)$, $\boldsymbol{j}^2$, $\tilde{L}_3$, J_3 and $\boldsymbol{J}^2$ ($\tilde{\boldsymbol{L}}$ being the space-fixed (left) angular momentum operator). The eigenfunctions of this spinor laplacian will then be labelled by j, L, J, N $(-L \leqslant N \leqslant L)$, $M(-J \leqslant M \leqslant J)$. Here N indicates the invariance of the hamiltonian (10) under left transformations of the three-sphere or, in the terminology of tops, under rotations of the top about a space-fixed axis, and M is simply the third component of $\boldsymbol{J}$ where $\boldsymbol{J} = \boldsymbol{j}+\boldsymbol{L}$. Eigenfunctions of Δ (and of any $\mathscr{H}_s$) in the q, μ representation are

$$\langle q, \mu | JLjMN\rangle = C\mathscr{D}^{L}_{NN'}(q)\begin{pmatrix} L & M & j \\ N' & J & \mu \end{pmatrix},$$

where C is a normalizing constant. This is a simple generalization of spin–orbit coupling (see, for instance, Landau and Lifshitz 1965, p 408) and as a general rule the states $|JLjMN\rangle$ may be employed in all the usual angular momentum calculations (eg Brink and Satchler 1962), the 'left' index N simply being a 'spectator'.

We intend to find the eigenvalues of the general hamiltonian (10) by expanding its eigenfunctions in terms of a suitable basic set and solving the secular equation with which standard theory then presents us. The complete commuting set for $\mathscr{H}$ is now reduced to just $\boldsymbol{j}^2$, $\boldsymbol{L}^2 = \tilde{\boldsymbol{L}}^2$ and L_3 and so the quantum numbers labelling such eigenfunctions will be j, L, N and of course E. The $|JLjMN\rangle$ may be used as a basis to obtain an expansion,

$$|jLNE\rangle = \sum_{JM} |JLjMN\rangle\langle JLjMN|jLNE\rangle,$$

but this will not be continued with because we know, by analogy with the (simple) case of the last section, that considerations of symmetry will simplify the problem. Therefore, the symmetry operators C^a, C^b, C^c which commute with the hamiltonian (10) are again introduced. As before, the effect of these may be found directly, to give

$$\begin{aligned} C^a|JLjMN\rangle &= \mathrm{e}^{-\mathrm{i}\pi(J+M)}|JLj-MN\rangle \\ C^b|JLjMN\rangle &= \mathrm{e}^{\mathrm{i}J\pi}|JLj-MN\rangle \\ C^c|JLjMN\rangle &= \mathrm{e}^{-\mathrm{i}M\pi}|JLjMN\rangle. \end{aligned} \tag{11}$$

We note that the states $|JLjMN\rangle$ transform under right operations as $\mathscr{D}^{J}_{NM}$ and this gives the above formulae. The situation is so similar to that of the last section that there is no need to repeat our arguments. All there is to remember is that $\mathscr{H}$ is no longer diagonal in J.

The proofs which show explicitly the required decomposition of the energy matrix, in a suitable basis, will now be given. The functions

$$|JM\gamma\alpha\rangle \equiv (2)^{-1/2}(|JLjMN\rangle + (-1)^{\gamma}|JLj-MN\rangle) \qquad (\alpha = j, N) \tag{12}$$

where $M \geqslant 0$ and $\gamma = 0, 1$ will be used, and the two cases of J integral and half-integral treated separately.

(i) *J integral*

From (11) and (12) we obtain

$$\begin{aligned} C^a|JM\gamma\alpha\rangle &= (-1)^{J+M+\gamma}|JM\gamma\alpha\rangle \\ C^b|JM\gamma\alpha\rangle &= (-1)^{J+\gamma}|JM\gamma\alpha\rangle \\ C^c|JM\gamma\alpha\rangle &= (-1)^{M}|JM\gamma\alpha\rangle. \end{aligned}$$

Using $C^a = (C^a)^{-1} = (C^a)^\dagger$ (and similarly for C^b, C^c) we obtain

$$\langle J'M'\gamma'\alpha|\mathscr{H}|JM\gamma\alpha\rangle$$
$$= \langle J'M'\gamma'\alpha|C^{b\dagger}\mathscr{H}C^b|JM\gamma\alpha\rangle = (-1)^{J+J'+\gamma+\gamma'}\langle J'M'\gamma'\alpha|\mathscr{H}|JM\gamma\alpha\rangle.$$

Therefore

$$(-1)^{\gamma+J} = (-1)^{\gamma'+J'}. \tag{13}$$

Using C^c in a similar manner, it can be shown that

$$(-1)^M = (-1)^{M'}. \tag{14}$$

(13) and (14) show that the hamiltonian splits into four blocks, each block being labelled by the parities of $(J+\gamma)$ and M, representing an irreducible representation of the group D_2 to which the operators C^a, C^b, C^c belong.

(ii) *J half-integral*

From (11) and (12) we obtain

$$C^a|JM\lambda\alpha\rangle = -\lambda\,\mathrm{e}^{-\mathrm{i}(J+M)\pi}|JM-\lambda\alpha\rangle$$
$$C^b|JM\lambda\alpha\rangle = \lambda\,\mathrm{e}^{\mathrm{i}J\pi}|JM\lambda\alpha\rangle$$
$$C^c|JM\lambda\alpha\rangle = \mathrm{e}^{-\mathrm{i}M\pi}|JM-\lambda\alpha\rangle,$$

where $\lambda \equiv (-1)^\gamma$, plus the corresponding adjoint expressions. Thus

$$C^a = C^cC^b = -C^bC^c, \quad \text{and} \quad C^{a\dagger} = -C^a = (C^a)^{-1}$$

etc. Thus,

$$\langle J'M'\lambda'\alpha|\mathscr{H}|JM\lambda\alpha\rangle = \langle J'M'\lambda'\alpha|C^{b\dagger}\mathscr{H}C^b|JM\lambda\alpha\rangle.$$

Therefore

$$(-1)^{J'+\gamma'} = (-1)^{J+\gamma}. \tag{15}$$

'Raising' and 'lowering' operators that act on λ may be defined, thus:

$$(e^{\mathrm{i}J\pi}C^c \pm C^a)|JM\lambda\alpha\rangle = \mathrm{e}^{\mathrm{i}(J-M)\pi}(1\pm\lambda)|JM-\lambda\alpha\rangle \tag{16}$$

and the corresponding adjoint expression. It is simple to show that

$$(\mathrm{e}^{-\mathrm{i}J'\pi}C^{c\dagger} \pm C^{a\dagger})(\mathrm{e}^{\mathrm{i}J\pi}C^c \pm C^a) = (\mathrm{e}^{\mathrm{i}J\pi}+\mathrm{e}^{\mathrm{i}J'\pi})(\mathrm{e}^{-\mathrm{i}J\pi} \mp C^b). \tag{17}$$

By calculating

$$\langle J'M'\lambda'\alpha|(\mathrm{e}^{-\mathrm{i}J'\pi}C^{c\dagger} \pm C^{a\dagger})\mathscr{H}(\mathrm{e}^{\mathrm{i}J\pi}C^c \pm C^a)|JM\lambda\alpha\rangle,$$

using (16) and (17), equating the two expressions obtained and then using (15), it can be shown that

$$\mathrm{e}^{-\mathrm{i}(M-M')\pi}\langle J'M'\lambda'\alpha|\mathscr{H}|JM\lambda\alpha\rangle = \langle J'M'-\lambda'\alpha|\mathscr{H}|JM-\lambda\alpha\rangle \tag{18}$$

which completes the proof.

Equations (15) and (18) show, firstly, that the hamiltonian splits into two blocks labelled by the parities of $(J+\gamma)$, and secondly that these two blocks have the same determinant. This leads to the twofold degeneracy that was earlier discussed, and justified on quite general grounds.

All that now remains to enable us to solve the eigenvalue equation

$$\mathscr{H}|jLNE\rangle = E|jLNE\rangle, \tag{19}$$

where $\mathscr{H}$ is given in (10), is to find the matrix elements

$$\mathscr{H}_{JJ',MM'} = \langle J'LjM'N|\mathscr{H}|JLjMN\rangle.$$

This is achieved by expressing $\mathscr{H}$ as a sum of tensor operators, as in equation (9), for the j, L and J terms in turn. Standard angular momentum theory may then be used to evaluate all the matrix elements $\langle J'LjM'N|T^K_k|JLjMN\rangle$ where T^K_k is a tensor (product) operator acting on J, j or L. A sample calculation of this sort will be found outlined in the appendix.

It is not possible to display the eigenvalues of the hamiltonian (10) in as convenient and general a way as that for the ordinary ($j = 0$) asymmetric top (see eg van Winter 1954, equation (61)) and so we present no numerical results. For given values of the coefficients l_i, m_i and n_i there is, however, no particular difficulty in diagonalizing the secular determinant, although its form is not so simple as for $j = 0$.

4. Discussion and conclusion

We have investigated the mathematical eigenvalue problem for the hamiltonian (10). The case of $j = 0$ is the ideal asymmetric top and the relevant complete set is essentially just that of the $\mathscr{D}^L_{MN}$—the hyperspherical harmonics, eigenfunctions of the Laplace–Beltrami–Casimir operator on SU(2). The generalization to nonzero j means, in effect, that the wavefunction becomes a spinor-valued quantity and the functions,

$$Y^{jML}_{\mu JN}(q) = (Y^{jML}_{JN})_\mu = C\mathscr{D}^L_{NN'}(q)\begin{pmatrix} L & M & j \\ N' & J & \mu \end{pmatrix},$$

which we can term 'spinor hyperspherical harmonics', form the relevant complete set. They are eigenfunctions of the spinor Laplace operator on SU(2) and are equivalent to the SO(4) harmonics, $D(I, A)_{LM,JN}$, discussed by Talman (1968) (See also Biedenharn 1961). A more detailed discussion of these functions and their relation to the spin harmonics of Lyubarskii (1960), Miller (1964) and Newman and Penrose (1966) will be presented elsewhere.

It may be of interest to note that, purely mathematically, the analysis is that of two coupled tops. The hamiltonian equivalent to (10) is then

$$\sum_i [l_i L_i^{(1)2} + m_i L_i^{(2)2} + n_i (L_i^{(1)} + L_i^{(2)})^2]$$

which can be taken as a purely differential operator acting on the six coordinates of the two tops.

The motivation presented in this paper for considering spinor functions is that they are needed if we want to discuss particles with spin in a Mixmaster universe. It is likely that there are other situations where the Y harmonics would be useful. There are, of course, many places where the $\mathscr{D}^L_{MN}$ occur. It might be worthwhile giving one or two. We have already mentioned the top. Another place is in the theory of SU(2) chiral dynamics where, if we ignore the spatial dependence of the fields, quantum chiral dynamics reduces to the theory of the spherical top (Dowker 1971, Dowker and Mayes 1971). The $\mathscr{D}^L_{MN}$ also occur in the theory of the hydrogen atom where they are, essentially, the momentum space wavefunctions, as is very well known. Whether it makes *physical* sense to generalize these situations to the spinor-valued case is open to discussion. The mathematical generalization is always possible.

5. Acknowledgments

One of us (DFP) wishes to thank the Science Research Council for the award of a research studentship.

Appendix. Calculation of $\langle J'LjM'N|T^K_k(\boldsymbol{L},\boldsymbol{L})|JLjMN\rangle$

The conventions are those of Brink and Satchler (1962). Use of the Wigner–Eckart theorem gives

$$\langle J'LjM'N|T^K_k(\boldsymbol{L},\boldsymbol{L})|JLjMN\rangle = \begin{pmatrix} J' & K & M \\ M' & k & J \end{pmatrix} \langle J'Lj\|T^K(\boldsymbol{L},\boldsymbol{L})\|JLj\rangle.$$

By Brink and Satchler (1962, equations (5.9) and (5.5)), we obtain

$$\langle J'Lj\|T^K(\boldsymbol{L},\boldsymbol{L})\|JLj\rangle$$
$$= [(2J+1)(2L+1)]^{1/2}(-1)^{L+J'+j+K}\begin{Bmatrix} J' & J & K \\ L & L & j \end{Bmatrix}\langle L\|T^K(\boldsymbol{L},\boldsymbol{L})\|L\rangle$$

with

$$\langle L\|T^K(\boldsymbol{L},\boldsymbol{L})\|L\rangle = \sum_{L'}(-1)^{K+L+L'}[(2L'+1)(2K+1)]^{1/2}\begin{Bmatrix} L & L & K \\ 1 & 1 & L' \end{Bmatrix}\langle L\|\boldsymbol{L}\|L'\rangle\langle L'\|\boldsymbol{L}\|L\rangle$$

and finally it is well known that

$$\langle L\|\boldsymbol{L}\|L'\rangle = \delta_{LL'}[L(L+1)]^{1/2}.$$

Thus everything is determined in terms of Racah coefficients and $3j$ symbols, which are extensively tabulated.

This and the similar results for $T^K_k(\boldsymbol{J},\boldsymbol{J})$ and $T^K_k(\boldsymbol{j},\boldsymbol{j})$ give a systematic way of evaluating the elements of the secular determinant of the hamiltonian (10).

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